Annali di Matematica 185, S187–S204 (2006)

Digital Object Identifier (DOI) 10.1007/s10231-004-0142-4

Urs Kirchgraber · Daniel Stoffer

Transversal homoclinic points of the Hénon map

Received: December 29, 2003; in final form: March 7, 2004
Published online: April 18, 2005 –

Abstract. Using shadowing techniques we prove that the Hénon map $H_{a,b}(x, y) = (a - x^2 + by, x)$ admits a transversal homoclinic point for a set of parameters which is not small. For the area and orientation preserving Hénon map (corresponding to $b = -1$) we prove that a transversal homoclinic point exists for $a \geq 0.265625$. Applying a computer-assisted version of our scheme we show that the result holds even for $a \geq -0.866$. This supports an old conjecture due to Devaney and Nitecki dating back to 1979, see [4], claiming that the Hénon map in the case $b = -1$ admits a transversal homoclinic point for $a > -1$.

Key words. Hénon map – transversal homoclinic point – shadowing

1. Introduction

One- and two-dimensional maps have played a remarkable role in the development of dynamical system theory over the last decades. Being slightly simpler to handle than systems of differential equations (at least in some respects), they exhibit the richest dynamical behaviour and therefore are a treasure and an important playground for experimentation for developing new ideas and concepts, to test methods, etc. In this paper we consider quadratic maps $P : \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{aligned} \overline{x} &= P_1(x, y) \\ \overline{y} &= P_2(x, y) \end{aligned} \tag{1}$$

with constant Jacobian determinant

$$\det \frac{\partial(\overline{x}, \overline{y})}{\partial(x, y)} = \text{constant}\,. \tag{2}$$

Polynomial maps (1) satisfying (2) are called *entire Cremona maps*. The inverse of (1) is also an entire Cremona map, see Engel [5,6]. The inverse of a quadratic Cremona map is again quadratic. There are various normal forms for quadratic Cremona maps in $\mathbb{R}^2$. One is Hénon's (see Hénon [11])

$$\begin{aligned} \overline{x} &= 1 - ax^2 + y \\ \overline{y} &= bx \end{aligned}$$

U. Kirchgraber, D. Stoffer: Department of Mathematics, ETH-Zürich, CH-8092 Zürich, Switzerland

or, after scaling,

$$\begin{aligned}\overline{x} &= a - x^2 + by \\ \overline{y} &= x\ .\end{aligned} \tag{3}$$

Note that the inverse of the Hénon map with parameters a and b is conjugate to the Hénon map with parameters a/b^2 and $1/b$, respectively. Therefore, without loss of generality, one may restrict oneself to parameters a, b satisfying $|b| \le 1$. According to Hénon, $a < -(1-b)^2/4$ implies that all orbits tend to infinity, and therefore there is no interesting dynamics in bounded domains. If $a > -(1-b)^2/4$, then the map (3) admits two fixed points. One of them, the point (x_1, x_1) with $x_1 = (b - 1 - \sqrt{(b-1)^2 + 4a})/2$, is hyperbolic for all $a > -(1-b)^2/4$. Let λ_1, λ_2, with $|\lambda_1| \le 1 \le |\lambda_2|$, denote the eigenvalues of the Jacobian at (x_1, x_1). We introduce new parameters $p := \lambda_1$ and $q := 1/\lambda_2$. Transforming the map (3) to new variables u, v

$$\begin{aligned}x &= x_1 + u/q \\ y &= x_1 + v/q\end{aligned}$$

and then writing again x, y instead of u, v the Hénon map is in a form appropriate for our purposes:

$$H: \quad \begin{aligned}\overline{x} &= \tfrac{1}{q}\left[(1 + pq)x - x^2 - py\right] \\ \overline{y} &= x\end{aligned}\ . \tag{4}$$

For completeness we state the relation between the parameter pairs a, b and p, q. Without loss of generality we may assume that $|p| \le q < 1$, $q > 0$. We have

$$\begin{aligned}a &= \frac{p + \frac{1}{q}}{2}\left(\frac{p + \frac{1}{q}}{2} - 1 - \frac{p}{q}\right) \\ b &= -\frac{p}{q}\end{aligned}$$

and

$$\begin{aligned}q &= \frac{1}{c + \sqrt{c^2 + b}}\ , \qquad c = (1 - b + \sqrt{(1-b)^2 + 4a})/2 \\ p &= -bq\ .\end{aligned}$$

The aim of this paper is twofold: First, we show that for a set of parameter values $p, q \in (-1, 1)$ (not small in size) the Hénon map admits chaotic behaviour. More precisely, we show that 0 admits a transversal homoclinic point for $|p| \le q \le 1/10$, see Theorem 5. Second, we consider the area and orientation preserving Hénon map, i.e. $p = q$, which in terms of a, b means $b = -1$. We prove that in this case there exists a transversal homoclinic point for $p = q \in (0, 1/4]$, corresponding to $b = -1$ and $a \ge 0.265625$, see Theorem 9. Moreover, with computer assistance we show the existence of transversal homoclinic points even for $a \ge -0.866360\ldots$.

Based on previous work, see Stoffer and Palmer [17], Stoffer and Kirchgraber [18], and Kirchgraber et al. [12], an improved scheme to establish chaotic behaviour using shadowing techniques was developed and applied in Kirchgraber and Stoffer [13]. While the application of this scheme to the restricted three-body problem in [13] was heavily based on computer assistance, the goal of the present paper is to show that the scheme is also very well suited to obtain analytical results.

To put the paper into perspective, we report on some known results. Since the seminal paper of Hénon [11] appeared, a large number of contributions have been made to better understand the behaviour of this map. We just mention a few. One of the deepest results is due to Benedicks and Carleson [1]. They show that for a approximately equal to 2 and sufficiently small b ($b = 0$ corresponds to the logistic map) the Hénon map admits a strange attractor. By constructing a horseshoe Devaney and Nitecki [4] show that for $|b| \leq 1, a > (5 + 2\sqrt{5})(1 + |b|)^2/4$ the nonwandering set of the Hénon map is topologically equivalent to the shift map of 2 symbols. In Sterling et al. [16] a large number of bifurcation results are reported. These authors also recover the main results of Devaney and Nitecki [4].

Several authors investigate the orientation and area preserving Hénon map, i.e. the case $b = -1$. Under this condition the Hénon map is reversible. In [4] Devaney and Nitecki conjecture that there exist transversal homoclinic points for all $a > -1$. By investigating the curvature of the stable and unstable manifolds Brown [2] proves that these manifolds indeed intersect transversally for $a > 0$. The strongest result in this respect we are aware of is due to Fontich [7]. By the graph transform method he proves that the stable and unstable manifolds intersect transversally for $a > -.3916$. Tovbis et al. [19] and Gelfreich and Sauzin [9] investigate exponentially small phenomena for $-1 < a << 0$.

Several authors investigate the case of the 'classical parameters' $a = 1.4$, $b = 0.3$. As far as we know all these results have been obtained with computer-assisted methods. Galias and Zgliczynski [8] prove the existence of several homoclinic and periodic orbits and rigorously estimate the topological entropy. Stoffer and Palmer [17] use shadowing techniques to establish chaotic behaviour. Coomes et al. [3] prove the existence of orbits homoclinic to a periodic orbit (in addition they also investigate the area preserving case $b = -1$).

The organization of the paper is as follows. In Section 2 we recall the basic shadowing result from [13], see also Pilyugin [15]. In the remarks following Theorem 1, and more so in Section 3, we exploit the special structure of the Hénon map in its form (4) yielding three auxiliary results (Lemmas 2–4) which are applied in Sections 4 and 5 to establish our main analytical results, see Theorems 5, 9. In the remarks at the very end we also report on some additional results obtained by rigorous numerics.

2. Shadowing

In this section we state a shadowing theorem proved in Stoffer and Kirchgraber [13], see also Pilyugin [15]. We first describe the general setting and give the necessary definitions. Let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^1. A sequence $w = (w_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ is called a *pseudo orbit of H with error $d = (d_n)_{n\in\mathbb{Z}} \in$*

$\ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ if $w_{n+1} - H(w_n) = d_n$ holds for $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ let Ω_n be an open and convex set containing w_n. An *R-shadowing orbit z of w* is a sequence $z = (z_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ with $z_n \in \Omega_n$, $z_{n+1} = H(z_n)$ and $|z_n - w_n| \leq R$ for $n \in \mathbb{Z}$.

We assume that the Jacobian DH does not vary too much in the set Ω_n. More precisely, we assume that there are (constant) 2×2-matrices A_n, with $\sup_{n\in\mathbb{Z}} |A_n| < \infty$ and such that $|DH - A_n|$ is small in the sense given below. We introduce the linear operator L and for $z \in \Omega := \prod_{n\in\mathbb{Z}} \Omega_n$ the linear operator Δ_z in $\ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ as follows:

$$\begin{aligned} L:&\ \ (L\zeta)_n = \zeta_{n+1} - A_n\zeta_n\ ;\\ \Delta_z:&\ \ (\Delta_z\zeta)_n = \big(DH(z_n) - A_n\big)\zeta_n\ . \end{aligned}$$

With these definitions we are able to state the shadowing result used in the sequel.

Theorem 1 (Shadowing Theorem). *Let $H \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and let δ_0, δ_1 be positive constants. For $n \in \mathbb{Z}$ let $\Omega_n \subset \mathbb{R}^2$ be open and convex. Let $w = (w_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$ with $w_n \in \Omega_n$ be a pseudo orbit of H with error $d \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$. Let the operators L and Δ_z be defined as above. Assume that L is invertible and that the following estimates are satisfied:*

$$\|L^{-1}d\| \leq \delta_0\,, \tag{5}$$

$$\|L^{-1}\Delta_z\| \leq \delta_1 \ <\ 1 \qquad \text{for all } z \in \Omega := \prod_{n\in\mathbb{Z}} \Omega_n\,, \tag{6}$$

$$\overline{B}_R(w_n) \subset \Omega_n \qquad \text{for} \quad R = \frac{\delta_0}{1-\delta_1}, \quad n \in \mathbb{Z}\,. \tag{7}$$

Then there is an R-shadowing orbit $z^* \in \Omega$ of w. z^* is the only orbit in Ω. Moreover, z^* is hyperbolic.

Remarks. 1. For our considerations the appropriate norm in $\mathbb{R}^2$ is the max-norm. Note that $\overline{B}_r(w_n)$ denotes the closed ball defined by

$$\begin{aligned} \overline{B}_r(w_n) &= \{z \mid |z - w_n| \leq r\}\\ &= \{z = (x, y)^T \mid |x - u_n| \leq r, |y - v_n| \leq r\}\,, \end{aligned}$$

where $w_n = (u_n, v_n)^T$. In $\ell^\infty(\mathbb{Z}, \mathbb{R}^i)$, $i = 1, 2$ the sup-norm is adopted as usual.

2. We shall apply Theorem 1 to the map H given in (4). Due to the special structure of H, we shall consider pseudo orbits $w = (w_n)_{n\in\mathbb{Z}}$, $w_n = (u_n, v_n)^T$ with $v_n = u_{n-1}$ for $n \in \mathbb{Z}$. Defining $u = (u_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$, we then have

$$\begin{aligned} ||w|| = \sup_n |w_n| \ &= \ \sup_n \max\{|u_n|, |v_n|\}\\ &= \max\{\sup_n |u_n|, \sup_n |v_n|\} \ = \ \sup_n |u_n| \ = \ ||u||\,. \end{aligned}$$

For the error d of the pseudo orbit w we get

$$\begin{aligned} d_n = w_{n+1} - H(w_n) &= \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} - H\begin{pmatrix} u_n \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{q}\big(qu_{n+1} - u_n(1+pq-u_n) + pu_{n-1}\big) \\ 0 \end{pmatrix} =: \frac{1}{q}\begin{pmatrix} r_n \\ 0 \end{pmatrix}. \end{aligned}$$

3. We shall set $\Omega_n = B_\rho(w_n) = \{z = (x,y)^T \mid |x-u_n| < \rho, |y-v_n| < \rho\}$ for some appropriate $\rho > 0$.
4. We set

$$A_n := DH(w_n) = DH\big((u_n, v_n)^T\big) = \begin{pmatrix} b_n/q & -p/q \\ 1 & 0 \end{pmatrix},$$

where $b_n := 1 + pq - 2u_n$. Then the operator Δ_z has a very simple form. Let $z = (z_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R}^2)$, $z_n = (x_n, y_n)^T \in B_\rho(w_n)$ be given. Then

$$\Delta_z : (\Delta_z\zeta)_n = D_n\zeta_n$$

with

$$D_n = \begin{pmatrix} -\frac{2}{q}(x_n - u_n) & 0 \\ 0 & 0 \end{pmatrix}.$$

3. Estimates for $||L^{-1}d||$ and $||L^{-1}\Delta_z||$

To estimate $||L^{-1}d||$ and $||L^{-1}\Delta_z||$, respectively, we solve $L\zeta = d$ and $L\zeta = \Delta_z\gamma$ with $|\gamma| \le 1$, respectively, for ζ and estimate $||\zeta||$.

We first consider $L\zeta = d$. According to Remarks 2 and 4 after Theorem 1 we find (setting $\zeta_n = (\xi_n, \eta_n)^T$)

$$L\zeta = d \Leftrightarrow \zeta_{n+1} - A_n\zeta_n = d_n, \quad n \in \mathbb{Z}$$

$$\Leftrightarrow \begin{array}{l} \xi_{n+1} - \frac{b_n}{q}\xi_n + \frac{p}{q}\eta_n = \frac{r_n}{q} \\ \eta_{n+1} - \xi_n = 0 \end{array}, \quad n \in \mathbb{Z}$$

$$\Leftrightarrow \begin{array}{r} -p\xi_{n-1} + b_n\xi_n - q\xi_{n+1} = -r_n \\ \eta_n = \xi_{n-1} \end{array}, \quad n \in \mathbb{Z}.$$

We introduce the linear operator $K : \xi \in \ell^\infty(\mathbb{Z}, \mathbb{R}) \mapsto K\xi \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ by setting

$$(K\xi)_n := -p\xi_{n-1} + b_n\xi_n - q\xi_{n+1} \tag{8}$$

with $b_n = 1 + pq - 2u_n$. One easily sees that K is invertible if and only if L is invertible. Since $\eta_n = \xi_{n-1}$ we further find

$$||L^{-1}d|| = ||\zeta|| = ||\xi|| = ||-K^{-1}r|| = ||K^{-1}r||,$$

where $r = (r_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$.

We now treat the equation $L\zeta = \Delta_z\gamma$ with $|\gamma_n| = |(\alpha_n, \beta_n)^T| \leq 1$. For the right–hand side we get, according to Remark 4, $(\Delta_z\gamma)_n = D_n\gamma_n = (-\frac{2}{q}(x_n - u_n)\alpha_n, 0)^T =: (\frac{e_n}{q}, 0)^T$, with $|e_n| \leq 2\rho$. Similarly as before we have

$$L\zeta = \Delta_z\gamma \Leftrightarrow \zeta_{n+1} - A_n\zeta_n = D_n\gamma_n, \quad n \in \mathbb{Z}$$

$$\Leftrightarrow \begin{array}{l} \xi_{n+1} - \frac{b_n}{q}\xi_n + \frac{p}{q}\eta_n = \frac{e_n}{q} \\ \eta_{n+1} - \quad \xi_n \qquad\quad = 0 \end{array}, \quad n \in \mathbb{Z}$$

$$\Leftrightarrow \begin{array}{r} -p\xi_{n-1} + b_n\xi_n - q\xi_{n+1} = -e_n \\ \eta_n = \xi_{n-1} \end{array}, \quad n \in \mathbb{Z}\,.$$

Setting $e = (e_n)_{n\in\mathbb{Z}}$ we obtain with the estimate $|e_n| \leq 2\rho$

$$||L^{-1}\Delta_z\gamma|| = ||\zeta|| = ||\xi|| = ||-K^{-1}e|| \leq ||K^{-1}||\;||e|| \leq 2\rho\,||K^{-1}||\,.$$

We summarize our results.

Lemma 2. *Let $w = (w_n)_{n\in\mathbb{Z}}$ with $w_n = (u_n, u_{n-1})^T$ be a pseudo orbit of the Hénon map H given in* (4). *Then w has error d with $d_n = \frac{1}{q}(r_n, 0)^T$, where*

$$r_n = pu_{n-1} - u_n(1 + pq - u_n) + qu_{n+1}\,.$$

The operator L is invertible if and only if K is invertible. Moreover, the following estimates hold:

$$\begin{aligned} ||L^{-1}d|| &= ||K^{-1}r|| \\ ||L^{-1}\Delta_z|| &\leq 2\,\rho\,||K^{-1}||\,. \end{aligned}$$

We now give a condition guaranteeing that Theorem 1 may be applied with an appropriate choice of ρ.

Lemma 3. *Let $w = (w_n)_{n\in\mathbb{Z}}$ with $w_n = (u_n, u_{n-1})^T$ be a pseudo orbit of the Hénon map H. Let $r = (r_n)_{n\in\mathbb{Z}}$ be given as in Lemma 2. If K is invertible and if*

$$8\,||K^{-1}r||\,||K^{-1}|| < 1 \tag{9}$$

holds, then Conditions (5)–(7) *of Theorem 1 are satisfied with $\delta_0 = ||K^{-1}r||$, $\delta_1 = 1/2$, and $\Omega_n = B_\rho(w_n)$, with $\rho = 1/(4\,||K^{-1}||)$.*

Proof. We have $||L^{-1}d|| = ||K^{-1}r|| = \delta_0$ by Lemma 2, and hence Condition (5) is satisfied. Lemma 2 implies $||L^{-1}\Delta_z|| \leq 2\,||K^{-1}||\,\rho = 1/2 = \delta_1$, verifying Condition (6). We now have $R = \delta_0/(1-\delta_1) = 2\,||K^{-1}r||$. By assumption we get $R < 1/(4\,||K^{-1}||) = \rho$, verifying Condition (7). □

To estimate $\|K^{-1}\|$, we use the following lemma.

Lemma 4. *Assume that the pseudo orbit $u = (u_n)_{n\in\mathbb{Z}}$ is homoclinic to 0 in the following sense: there is N such that $u_n = 0$ if $|n| \geq N$. Let K be defined by* (8) *with* $b_n = 1 + pq - 2u_n$. *Assume that for* $i \in \mathbb{Z}$ *the equation*

$$K\xi = t^{(i)} ,$$

with $t^{(i)} = (t_n^{(i)})_{n\in\mathbb{Z}}$, $t_n^{(i)} = \delta_{ni}$ (δ_{ni} *the Kronecker symbol), has a solution* $\xi^{(i)} = \left(\xi_n^{(i)}\right)_{n\in\mathbb{Z}}$ *and assume*

$$k := \sup_n \sum_{i\in\mathbb{Z}} \left|\xi_n^{(i)}\right| < \infty .$$

Then the operator K *is invertible and* $\|K^{-1}\| = k$ *holds.*

Proof. Let $c = (c_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ and assume $\|c\| \le 1$, i.e. $|c_n| \le 1$, for all $n \in \mathbb{Z}$. Then, since $k < \infty$ by assumption, the sequence $a = (a_n)_{n\in\mathbb{Z}}$ given by $a_n = \sum_{i\in\mathbb{Z}} c_i \xi_n^{(i)}$ is well defined and is an element of $\ell^\infty(\mathbb{Z}, \mathbb{R})$ and $\|a\| \le k$. It is easy to verify that $(Ka)_n = c_n$ and therefore $Ka = c$. This shows that if K is invertible, then $\|K^{-1}\| \le k$. Given $\varepsilon > 0$, there is n such that $\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| \ge k - \varepsilon$. Then for $c_i = \mathrm{sign}(\xi_n^{(i)})$ we have $a_n = \sum_{i\in\mathbb{Z}} c_i \xi_n^{(i)} \ge k - \varepsilon$. Thus if K^{-1} exists, then $\|K^{-1}\| = k$. Because $Kx = c$, $\|c\| \le 1$ is solvable as shown above, it remains to prove that K is injective in order for K^{-1} to exist. By suitably combining the equations $(K\xi)_n = t_n$ for $n \le -N$ and $n \ge N$, respectively, it is easy to see that the system $K\xi = t$ is equivalent to the following systems of equations:

$$\xi_n - q\xi_{n+1} = \cdots + p^2 t_{n-2} + p t_{n-1} + t_n, \quad n \le -N , \tag{10}$$

$$-p\xi_{n-1} + b_n \xi_n - q\xi_{n+1} = t_n, \quad -N < n < N , \tag{11}$$

$$-p\xi_{n-1} + \xi_n = t_n + q t_{n+1} + q^2 t_{n+2} + \ldots, \quad n \ge N . \tag{12}$$

The equations for $-N \le n \le N$ build a system of $2N + 1$ equations with $2N + 1$ unknowns. We will refer to these equations as the 'inner system'. The assumption of the lemma implies that this system has a solution for every right–hand side. It follows that the solution is unique. Equations (10) and (12) for $n < -N$ and $n > N$, respectively, determine the remaining ξ_n uniquely. Hence K is indeed injective. □

4. A simple pseudo orbit

In this section we show that for $0 \le |p| \le q \le 1/10$ the scheme derived in the previous sections may be applied to the very simple pseudo orbit $w = (w_n)_{n\in\mathbb{Z}}$, $w_n = (u_n, u_{n-1})^T$, with

$$u_n = \begin{cases} 1, & n = 0 \\ 0, & \text{else} \end{cases} .$$

The numbers b_n and r_n are easily found to be

$$b_n = \begin{cases} -1 + pq, & n = 0 \\ 1 + pq, & \text{else} \end{cases} , \qquad r_n = \begin{cases} q, & n = -1 \\ -pq, & n = 0 \\ p, & n = 1 \\ 0, & \text{else} \end{cases} .$$

The goal is to verify Condition (9) of Lemma 3. For this simple pseudo orbit the quantities $||K^{-1}r||$ and $||K^{-1}||$ will be computed explicitly.

(a) We first determine $||K^{-1}||$ using Lemma 4. Here we have $N = 1$. We have to solve the systems $K\xi = t$ for $t = t^{(i)}$, $i \in \mathbb{Z}$. As seen in the proof of Lemma 4, these systems are equivalent to (10)–(12). Adding the p-fold of (10) for $n = -1$ and the q-fold of (12) for $n = 1$ to (11) with $n = 0$ we get the following system of equations:

$$\begin{aligned} \xi_n^{(i)} - q\xi_{n+1}^{(i)} &= s_n^{(i)}, \qquad && n \le -1\,, \\ -(1+pq)\xi_0^{(i)} &= s_0^{(i)}, && \\ -p\xi_{n-1}^{(i)} + \xi_n^{(i)} &= s_n^{(i)}, && n \ge 1\,. \end{aligned} \tag{13}$$

The right-hand sides $s_n^{(i)}$ are given by

$$s_n^{(i)} = \begin{cases} p^{n-i}, & i \le n \le 0 \\ q^{i-n}, & 0 \le n \le i \\ 0, & \text{else}\,. \end{cases}$$

It is not hard to verify that the solution of (13) is given by the following formulas. For $i \ge 0$ we have

$$\xi_n^{(i)} = \begin{cases} q^{i-n}\dfrac{-1}{1+pq}, & n \le 0 \le i \\ q^{i-n}\dfrac{1+pq-2(pq)^n}{1-(pq)^2}, & 0 \le n \le i \\ p^{n-i}\dfrac{1+pq-2(pq)^i}{1-(pq)^2}, & 0 \le i \le n \end{cases},$$

while for $i \le 0$ the following formulas hold:

$$\xi_n^{(i)} = \begin{cases} p^{n-i}\dfrac{-1}{1+pq}, & i \le 0 \le n \\ p^{n-i}\dfrac{1+pq-2(pq)^{|n|}}{1-(pq)^2}, & i \le n \le 0 \\ q^{i-n}\dfrac{1+pq-2(pq)^{|i|}}{1-(pq)^2}, & n \le i \le 0 \end{cases}.$$

We determine $\|K^{-1}\|$ according to Lemma 4. First we consider the case $0 \le p \le q < 1$. We find

$$\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| = \frac{1}{(1-p)(1-q)} - \frac{C_n}{(1-p)(1-q)(1+pq)}$$

with

$$C_n = \begin{cases} 2pq, & n = 0 \\ 2p^n q, & n > 0 \\ 2pq^{-n}, & n < 0 \end{cases}.$$

For $0 \le p \le q < 1$ we thus get $\|K^{-1}\| = \frac{1}{(1-p)(1-q)}$. We next consider the case $0 \le -p \le q < 1/2$. For $n = -1, 0, 1$ one finds $\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| = \frac{1}{(1-|p|)(1-q)}$. For all other n one finds $\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| = \frac{1}{(1-|p|)(1-q)} - B_n$ with $B_n > 0$. It follows that

(14) $$||K^{-1}|| = \frac{1}{(1-|p|)(1-q)} \,.$$

(b) To determine $||K^{-1}r||$, we solve $K\xi = r$ for ξ and determine $||\xi||$. The equation $K\xi = r$ is equivalent to the system (10)–(12) with t_n replaced by r_n. Again by adding the p-fold of (10) for $n = -1$ and the q-fold of (12) for $n = 1$ to (11) with $n = 0$ we get the following system of equations:

$$\begin{aligned} \xi_n - q\xi_{n+1} &= q\,\delta_{-1,n}, && n \le -1\,, \\ -(1+pq)\xi_0 &= pq, \\ -p\xi_{n-1} + \xi_n &= p\,\delta_{1,n}, && n \ge 1\,. \end{aligned}$$

It is easy to verify that the numbers

$$\xi_n = \begin{cases} \dfrac{p^n}{1+pq}, & n > 0 \\ \dfrac{-pq}{1+pq}, & n = 0 \\ \dfrac{q^{-n}}{1+pq}, & n < 0 \end{cases}$$

solve this system of equations. This leads to

(15) $$||K^{-1}r|| = ||\xi|| = \sup_n\{|\xi_n|\} = \frac{q}{1+pq} \,.$$

We conclude:

Theorem 5. *The Hénon map H, cf.* (4)*, admits a transversal homoclinic point for all parameters p, q with* $0 < |p| \le q \le 1/10$*. In terms of the classical parameters a, b this means:* $|b| \in (0, 1]$ *and* $a \ge 20 + \frac{91}{20}b - \frac{19}{400}b^2$.

Proof. According to Lemma 3 we have to verify that Condition (9) is satisfied. Equations (14) and (15) imply that

$$\text{Left-hand side} = 8\,\frac{q}{1+pq}\,\frac{1}{(1-|p|)(1-q)} \le 8\,\frac{1}{10}\,\frac{100}{99}\,\frac{10}{9}\,\frac{10}{9} = \frac{8000}{8019} < 1\,.$$

It follows that Theorem 1 may be applied. From the uniqueness of the shadowing orbits it follows that the projection $\Pi : w \mapsto z_0^*$ is continuous. Applying the shift map σ to the pseudo orbit we get $\Pi(\sigma^n w) = z_n^*$. Since $\lim_{n\to\pm\infty} \sigma^n w = 0 = (0)_{n\in\mathbb{Z}}$, one finds by taking the limit $n \to \pm\infty$ that $\lim_{n\to\pm\infty} z_n^* = 0$. This means that the shadowing orbit z is homoclinic to the hyperbolic fixed point 0. By Theorem 1 it is also hyperbolic. According to Palmer [14], hyperbolicity is equivalent to transversality. $\square$

5. The area preserving Hénon map: improved results

In this section we prove that the area and orientation preserving Hénon map H admits a transversal homoclinic point for $p = q \leq 1/4$, which, in terms of the classical parameters, corresponds to $b = -1$ and $a \geq 0.265625$.

The reasoning closely follows the lines of the previous section. We therefore only present the key computations.

We choose the following pseudo orbit $w = (w_n)_{n\in\mathbb{Z}}$ with $w_n = (u_n, u_{n-1})^T$ and

n	≤ -4	-3	-2	-1	0	1	2	≥ 3
u_n	0	p^2	p	$1-p$	$1-p$	p	p^2	0

A brief computation yields

$$(16)\quad b_n = \begin{cases} -(1-2p-p^2), & n=0 \\ 1-2p+p^2, & n=1 \\ 1-p^2, & n=2 \\ 1+p^2, & n\geq 3 \\ b_{-n-1}, & n<0 \end{cases}, \qquad r_n = \begin{cases} p^3, & n=-4,-1,0,3 \\ 0, & \text{else} \end{cases}.$$

This implies that the pseudo orbit w has error d of order p^2.

The goal is to estimate $||K^{-1}||$ and $||K^{-1}r||$ and to eventually verify that Condition (9) of Lemma 3 is satisfied for $p \leq 1/4$. We begin with $||K^{-1}||$.

5.1. *An estimate for* $||K^{-1}||$

Let us reflect on the strategy for a moment. To find an estimate for $||K^{-1}||$, we have to estimate

$$\sup_n \sum_{i\in\mathbb{Z}} \left|\xi_n^{(i)}\right|$$

according to Lemma 4. Therefore we need estimates for $\xi_n^{(i)}$. First we will show how one might compute the $\xi_n^{(i)}$ in principle. The result is the infinite system of linear equations (17) and (21). Then, in order to obtain the desired estimates for $\xi_n^{(i)}$, we derive two lemmas. In Lemma 6 we give bounds for the constants c_0, c_1, c_2 which appear in the diagonal of (21), while in Lemma 7 we estimate $s_n^{(i)}$, i.e. the right-hand sides of (17) and (21). Lemma 8 finally is the key step to estimate $||K^{-1}||$. Equation (24) is the result.

Following Lemma 4 we consider the systems

$$K\xi^{(i)} = t^{(i)}, \quad \text{where} \quad t^{(i)} = \left(t_n^{(i)}\right)_{n\in\mathbb{Z}} \quad \text{with} \quad t_n^{(i)} = \begin{cases} 1, & \text{if } n=i \\ 0, & \text{else} \end{cases} \qquad i\in\mathbb{Z}$$

and estimate $||K^{-1}||$ according to Lemma 4. In analogy to (10) and (12) we have

$$(17)\qquad \begin{aligned} \xi_n^{(i)} - p\xi_{n+1}^{(i)} &= t_n^{(i)} + pt_{n-1}^{(i)} + p^2 t_{n-2}^{(i)} + \ldots =: s_n^{(i)}, \qquad & n\leq -4 \\ -p\xi_{n-1}^{(i)} + \xi_n^{(i)} &= t_n^{(i)} + pt_{n+1}^{(i)} + p^2 t_{n+2}^{(i)} + \ldots =: s_n^{(i)}, \qquad & n\geq 3\,. \end{aligned}$$

The following inner system is obtained from (17) and (11):

$$
\begin{array}{cccccccc|c}
\xi_{-4}^{(i)} & \xi_{-3}^{(i)} & \xi_{-2}^{(i)} & \xi_{-1}^{(i)} & \xi_{0}^{(i)} & \xi_{1}^{(i)} & \xi_{2}^{(i)} & \xi_{3}^{(i)} & \\
\hline
1 & -p & & & & & & & s_{-4}^{(i)} \\
-p & b_2 & -p & & & & & & t_{-3}^{(i)} \\
 & -p & b_1 & -p & & & & & t_{-2}^{(i)} \\
 & & -p & b_0 & -p & & & & t_{-1}^{(i)} \\
 & & & -p & b_0 & -p & & & t_{0}^{(i)} \\
 & & & & -p & b_1 & -p & & t_{1}^{(i)} \\
 & & & & & -p & b_2 & -p & t_{2}^{(i)} \\
 & & & & & & -p & 1 & s_{3}^{(i)}
\end{array}
$$

To simplify the inner system, we perform symmetric Gauss elimination. We add the p–fold of the first equation to the second one and the p–fold of the eighth equation to the seventh one, respectively, and define

$$c_2 := b_2 - p^2, \qquad s_{-3}^{(i)} := t_{-3}^{(i)} + p s_{-4}^{(i)}, \qquad s_{2}^{(i)} := t_{2}^{(i)} + p s_{3}^{(i)} . \tag{18}$$

Then we add the p/c_2–fold of the second equation to the third one and the p/c_2–fold of the seventh equation to the sixth one, respectively, and define

$$c_1 := b_1 - \frac{p^2}{c_2}, \qquad s_{-2}^{(i)} := t_{-2}^{(i)} + \frac{p}{c_2} s_{-3}^{(i)}, \qquad s_{1}^{(i)} := t_{1}^{(i)} + \frac{p}{c_2} s_{2}^{(i)} . \tag{19}$$

Finally we add the p/c_1–fold of the third equation to the fourth one and the p/c_1–fold of the sixth equation to the fifth one, respectively, and define

$$c_0 := b_0 - \frac{p^2}{c_1}, \qquad s_{-1}^{(i)} := t_{-1}^{(i)} + \frac{p}{c_1} s_{-2}^{(i)}, \qquad s_{0}^{(i)} := t_{0}^{(i)} + \frac{p}{c_1} s_{1}^{(i)} . \tag{20}$$

Eventually we obtain the following linear system:

$$
\begin{array}{cccccccc|c}
\xi_{-4}^{(i)} & \xi_{-3}^{(i)} & \xi_{-2}^{(i)} & \xi_{-1}^{(i)} & \xi_{0}^{(i)} & \xi_{1}^{(i)} & \xi_{2}^{(i)} & \xi_{3}^{(i)} & \\
\hline
1 & -p & & & & & & & s_{-4}^{(i)} \\
 & c_2 & -p & & & & & & s_{-3}^{(i)} \\
 & & c_1 & -p & & & & & s_{-2}^{(i)} \\
 & & & c_0 & -p & & & & s_{-1}^{(i)} \\
 & & & -p & c_0 & & & & s_{0}^{(i)} \\
 & & & & -p & c_1 & & & s_{1}^{(i)} \\
 & & & & & -p & c_2 & & s_{2}^{(i)} \\
 & & & & & & -p & 1 & s_{3}^{(i)}
\end{array}
\tag{21}
$$

Before exploiting the system (17), (21) further, we derive estimates for the numbers c_i and the right-hand sides $s_n^{(i)}$.

Lemma 6. *For $p \in [0, 1/4]$ the following estimates hold:*

$$
\begin{aligned}
&(i) &\frac{7}{8} &\le \; c_2 \; = 1 - 2p^2\,, \\
&(ii) &\frac{55}{112} &\le \; c_1 \; \le 1 - \frac{13}{7}p - \frac{5}{7}p^2\,, \\
&(iii) &\frac{497}{880} &\le -c_0 \le 1\,.
\end{aligned}
$$

Proof. (i) By definition $c_2 = 1 - 2p^2$ holds and hence c_2 is monotonically decreasing with respect to p. Claim (i) follows.

(ii) Using (i) we get $c_1 = (1-p)^2 - p^2/c_2 \ge \frac{9}{16} - \frac{1}{16}\frac{8}{7} = \frac{55}{112}$.

To establish the upper bound we have to verify that

$$1 - 2p + p^2 - \frac{p^2}{1-2p^2} \le 1 - \frac{13}{7}p - \frac{5}{7}p^2\,.$$

We subtract $1 - 2p + p^2$ and then multiply by $1 - 2p^2$. We get the equivalent inequality

$$-p^2 \le \left(\frac{1}{7}p - \frac{12}{7}p^2\right)(1-2p^2)\,.$$

We multiply by $7/p$, add $7p$, and get the equivalent inequality

$$0 \le 1 - 5p - 2p^2 + 24p^3 = (1-4p)(1-p-6p^2)\,,$$

which is obviously satisfied for $p \in [0, 1/4]$.

(iii) We first show

$$1 - 2p + \frac{p^2}{c_1} \ge \frac{69}{110}\,. \tag{22}$$

This inequality is equivalent to

$$\left(\frac{41}{110} - 2p\right)c_1 + p^2 \ge 0\,.$$

This inequality certainly holds if $\frac{41}{110} - 2p \ge 0$. If not, it is – in view of (ii) – sufficient to show that

$$\left(\frac{41}{110} - 2p\right)\left(1 - \frac{13}{7}p - \frac{5}{7}p^2\right) + p^2 \ge 0\,.$$

We multiply the last inequality by 770 and successively find the following equivalent inequalities

$$
\begin{aligned}
(41 - 220p)(7 - 13p - 5p^2) + 770p^2 &\ge 0\,, \\
287 - 2073p + 3425p^2 + 1100p^3 &\ge 0\,, \\
(1-4p)(287 - 925p - 275p^2) &\ge 0\,.
\end{aligned}
$$

Since the last inequality is obviously satisfied, (22) indeed holds. By definition of c_0 we have

$$-c_0 = 1 - 2p - p^2 + \frac{p^2}{c_1} \geq \min\left\{1 - 2p + \frac{p^2}{c_1}\right\} + \min\{-p^2\}$$
$$\geq \frac{69}{110} - \frac{1}{16} = \frac{497}{880} .$$

Using (ii) we find the upper estimate for $-c_0$

$$-c_0 \leq 1 - 2p - p^2 + \frac{112}{55}p^2 \leq 1 - 2p(1-p) \leq 1 .$$

□

We now estimate the right-hand sides of the system (17), (21).

Lemma 7. *For* $p \in [0, 1/4]$ *the estimates* $0 \leq s_n^{(i)} \leq \hat{s}_n^{(i)}$ *hold* $i, n \in \mathbb{Z}$ *with*

$\hat{s}_n^{(i)}$	$i<0$	$i=0$	$i=1$	$i \geq 2$
$n=0$	0	1	$\frac{28}{55}$	$\frac{28}{55} \cdot \frac{2}{7} \cdot \left(\frac{1}{4}\right)^{i-2}$
$n=1$	0	0	1	$\frac{2}{7} \cdot \left(\frac{1}{4}\right)^{i-2}$
$n=2$	0	0	0	$\left(\frac{1}{4}\right)^{i-2}$

For $n \geq 3$

$$\hat{s}_n^{(i)} = \begin{cases} \left(\frac{1}{4}\right)^{i-n}, & i \geq n \\ 0, & i < n \end{cases} .$$

Moreover, the symmetry relations

$$s_{-n}^{(-i)} = s_{n-1}^{(i-1)}, \qquad \hat{s}_{-n}^{(-i)} = \hat{s}_{n-1}^{(i-1)}, \qquad \xi_{-n}^{(-i)} = \xi_{n-1}^{(i-1)} \tag{23}$$

hold for all $i, n \in \mathbb{Z}$.

Proof. The statement for $n \geq 3$ follows from (17).

The statement for $n = 2$ follows from the definition in (18).

The statement for $n = 1$ follows from the definition in (19) and estimate (i) in Lemma 6.

The statement for $n = 0$ follows from definition in (20) and estimate (ii) in Lemma 6.

The symmetry relations follow from (17)–(20), from $t_{-n}^{(-i)} = t_{n-1}^{(i-1)}$, and from the symmetry of system (21). □

In view of Lemma 4 we eventually estimate $\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}|$ for $n \in \mathbb{Z}$.

Lemma 8. *For $p \in [0, 1/4]$ the following estimates hold:*

$$(i) \qquad \textstyle\sum_{i\in\mathbb{Z}} \left|\xi_0^{(i)}\right| \le K_0 := \dfrac{4496}{831} = 5.41\ldots .$$

$$(ii) \qquad \textstyle\sum_{i\in\mathbb{Z}} \left|\xi_1^{(i)}\right| \le K_1 := \dfrac{28}{5} = 5.6 .$$

$$(iii) \qquad \textstyle\sum_{i\in\mathbb{Z}} \left|\xi_2^{(i)}\right| \le K_2 := \dfrac{328}{105} < K_1 .$$

$$(iv) \qquad \textstyle\sum_{i\in\mathbb{Z}} \left|\xi_n^{(i)}\right| \le K_n := \dfrac{K_{n-1}}{4} + \dfrac{4}{3} < K_{n-1} \qquad \text{for} \quad n \ge 3 .$$

$$(v) \qquad \textstyle\sum_{i\in\mathbb{Z}} \left|\xi_{-n}^{(i)}\right| \le K_{n-1} \qquad \text{for} \quad n > 0 .$$

Proof. (i) Since $0 \le p < -c_0$ (as follows from Lemma 6), we find from the two central equations of (21) that $\xi_0^{(i)} \le 0$, $\xi_{-1}^{(i)} \ge 0$ for $i \ge 0$ (as follows from $s_{-1}^{(i)} = s_0^{(-i-1)} = 0$ and $s_0^{(i)} \ge 0$ for $i \ge 0$). We conclude by taking advantage of (23):

$$\left|\xi_0^{(i)}\right| + \left|\xi_0^{(-i-1)}\right| = \left|\xi_0^{(i)}\right| + \left|\xi_{-1}^{(i)}\right| = \xi_{-1}^{(i)} - \xi_0^{(i)} .$$

Taking the difference of the two central equations of (21) we get $(-c_0 - p)(\xi_{-1}^{(i)} - \xi_0^{(i)}) = s_0^{(i)}$ and therefore

$$\left|\xi_0^{(i)}\right| + \left|\xi_0^{(-i-1)}\right| = \frac{s_0^{(i)}}{|c_0| - p} .$$

Taking the sum for $i \ge 0$ leads to

$$\sum_{i\in\mathbb{Z}} \left|\xi_0^{(i)}\right| = \frac{1}{|c_0| - p} \sum_{i\ge 0} s_0^{(i)} .$$

Lemmas 6 and 7 yield

$$|c_0| - p \ge \frac{497}{880} - \frac{1}{4} = \frac{277}{880}$$

and

$$\sum_{i\ge 0} s_0^{(i)} \le \sum_{i\ge 0} \hat{s}_0^{(i)} = 1 + \frac{28}{55}\left(1 + \frac{2}{7}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots\right)\right) = \frac{281}{165} .$$

Combining the above formulas we eventually find

$$\sum_{i\in\mathbb{Z}} \left|\xi_0^{(i)}\right| \le \frac{880}{277}\,\frac{281}{165} = \frac{4496}{831} .$$

(ii) From (21) we get

$$\left|\xi_1^{(i)}\right| \le \left|\frac{1}{c_1}(s_1^{(i)} + p\xi_0^{(i)})\right| \le \frac{1}{c_1}\left(\hat{s}_1^{(i)} + \frac{1}{4}\left|\xi_0^{(i)}\right|\right).$$

Taking the sum for $i \in \mathbb{Z}$ gives with $\sum_{i\in\mathbb{Z}} \hat{s}_1^{(i)} = \sum_{i\geq 0} \hat{s}_1^{(i)} = 1 + \frac{2}{7}(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots) = \frac{29}{21}$

$$\sum_{i\in\mathbb{Z}} |\xi_1^{(i)}| \leq \frac{1}{c_1}\left(\sum_{i\in\mathbb{Z}} \hat{s}_1^{(i)} + \frac{K_0}{4}\right) \leq \frac{112}{55}\left(\frac{29}{21} + \frac{1124}{831}\right)$$
$$= \frac{254416}{45705} = 5.566\cdots < \frac{28}{5} .$$

(iii) As in (ii) we find

$$\sum_{i\in\mathbb{Z}} |\xi_2^{(i)}| \leq \frac{1}{c_2}\left(\frac{K_1}{4} + \sum_{i\geq 0} \hat{s}_2^{(i)}\right) .$$

Since $\sum_{i\geq 0} \hat{s}_n^{(i)} = 1/(1-p) \leq 4/3$, for $n \geq 2$ we get

$$\sum_{i\in\mathbb{Z}} |\xi_2^{(i)}| \leq \frac{8}{7}\left(\frac{7}{5} + \frac{4}{3}\right) = \frac{328}{105} .$$

(iv) Similarly as in (iii) we find for $n \geq 3$

$$\sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| \leq \frac{K_{n-1}}{4} + \sum_{i\geq 0} \hat{s}_n^{(i)} = \frac{K_{n-1}}{4} + \frac{4}{3} .$$

Since $K_2 > \frac{16}{9}$, the sequence $(K_n)_{n\geq 2}$ is geometrically decreasing towards the limit 16/9.

(v) The claim follows from (23). □

As a consequence of Lemmas 4 and 8 we get the following estimate for $||K^{-1}||$:

(24) $$||K^{-1}|| \leq \sup_n \sum_{i\in\mathbb{Z}} |\xi_n^{(i)}| \leq \frac{28}{5} \qquad \text{for} \quad p \in [0, 1/4] .$$

Remark. A more careful estimate in assertion (ii) of Lemma 8 reveals that in fact $\sum_{i\in\mathbb{Z}} |\xi_1^{(i)}| \leq \sum_{i\in\mathbb{Z}} |\xi_0^{(i)}|$ holds and that therefore $||K^{-1}|| \leq K_0$ for $p \in [0, 1/4]$. We do not need this sharper estimate, however.

5.2. An estimate for $||K^{-1}r||$

We now estimate $||K^{-1}r||$, i.e. we solve $K\xi = r$ with $r = p^3(t^{(-4)} + t^{(-1)} + t^{(0)} + t^{(3)})$, see (16), and estimate $||\xi||$. The system corresponding to (21) now reads

ξ_{-4}	ξ_{-3}	ξ_{-2}	ξ_{-1}	ξ_0	ξ_1	ξ_2	ξ_3	
1	$-p$							s_{-4}
	c_2	$-p$						s_{-3}
		c_1	$-p$					s_{-2}
			c_0	$-p$				s_{-1}
			$-p$	c_0				s_0
				$-p$	c_1			s_1
					$-p$	c_2		s_2
						$-p$	1	s_3

with $s_n = p^3(s_n^{(-4)} + s_n^{(-1)} + s_n^{(0)} + s_n^{(3)})$ and hence $s_{-n} = s_{n-1}$. The symmetry of the system implies $\xi_{-n} = \xi_{n-1}$ (take the difference of the two central equations). Taking the sum of the two central equations we find

$$(c_0 - p)(\xi_{-1} + \xi_0) = 2(c_0 - p)\xi_0 = 2p^3\big(s_0^{(0)} + s_0^{(3)}\big) = 2s_0$$

and therefore

$$-\xi_0 = |\xi_0| = \frac{s_0}{|c_0| + p} = \frac{p}{|c_0| + p}\, p^2\big(s_0^{(0)} + s_0^{(3)}\big) .$$

Note that $p/(|c_0| + p)$ and p^2 are monotonically increasing with respect to p and so is their product. Thus we find

$$|\xi_0| \le \frac{220}{717}\,\frac{1}{16}\,\frac{57}{55} = \frac{19}{956} = 0.01987\ldots .$$

Moreover, we have

$$\begin{aligned}|\xi_1| &= \frac{1}{c_1}\,|s_1 + p\xi_0| \le \frac{112}{55}\left(\frac{1}{14}\,\frac{1}{64} + \frac{1}{4}\,\frac{19}{956}\right) < \frac{7}{55}\left(\frac{1}{56} + \frac{1}{12}\right) < \frac{1}{64}\\ &= 0.015\ldots ,\end{aligned}$$

$$|\xi_2| = \frac{1}{c_2}\,|s_2 + p\xi_1| \le \frac{8}{7}\left(\frac{1}{4}\,\frac{1}{64} + \frac{1}{4}\,\frac{1}{64}\right) = \frac{1}{112} ,$$

$$|\xi_3| = |s_3 + p\xi_2| \le \frac{1}{64} + \frac{1}{4}\,\frac{1}{112} = \frac{1}{56} = 0.017\ldots ,$$

$$|\xi_{n+1}| = p|\xi_n|, \qquad n \ge 3 .$$

We therefore conclude

$$||K^{-1}r|| = \sup_n |\xi_n| \le \frac{19}{956} \qquad \text{for} \quad p \in [0, 1/4] . \tag{25}$$

5.3. Application of Theorem 1

We now easily get the following result.

Theorem 9. *The area and orientation preserving Hénon map H, cf.* (4)*, admits a transversal homoclinic point for all parameters p, q with $p = q \in$* (0, 1/4]. *In terms of the classical parameters a, b, this corresponds to $b = -1$ and $a \ge \frac{17}{64} =$* 0.265625.

Proof. According to Lemma 3 we have to verify that Condition (9) is satisfied. Equations (24) and (25) imply that

$$\text{Left–hand side } \le 8\,\frac{19}{956}\,\frac{28}{5} = \frac{1064}{1195} < 1 .$$

It follows that Theorem 1 applies. As in the proof of Theorem 5 we conclude that the shadowing orbit is homoclinic to 0 and transversal. □

Remarks. 1. Theorem 9 considerably generalizes one of the results of Coomes et al. [3]. With computer assistance they prove the existence of a transversal homoclinic point for $b = -1$ and $a = 1$.

2. Computer experiments indicate that for the pseudo orbit chosen above Condition (9) is in fact satisfied for $p = q < p_0 = 0.256637471\ldots$. This corresponds to $b = -1$ and $a > a_0 = 0.159050966\ldots$.
3. Using the graph transforms method Fontich [7] has shown that the stable and unstable manifolds of the fixed point 0 intersect transversally for $a \geq -0.3916$.
4. The tools developed in this paper lend themselves to a computer–assisted aproach. A simple version is as follows. For a particular value of the parameter $p = q = p_0$ a homoclinic pseudo orbit is computed numerically. To verify (9) one first computes an estimate for $\|K^{-1}r\|$ for parameters $p = q$ in a small neighbourhood of p_0 using interval arithmetic. Then $\|K^{-1}\|$ is estimated similarly. In the next step, p_0 is slightly enlarged and the procedure is repeated such that eventually a 'large' interval is covered.
   This way we were able to establish the existence of a transversal homoclinic point for $b = -1$, $a \geq -0.866360\ldots$. Note that this result supports the conjecture of Devaney and Nitecki [4] mentioned in the introduction. The details of a much more elaborate version are in preparation.

References

1. Benedicks, M., Carleson, L.: The dynamics of the Hénon map. Ann. Math. **133**, 73–169 (1991)
2. Brown, R.: Horseshoes in the measure–preserving Hénon map. Ergodic Theory Dyn. Syst. **15**, 1045–1059 (1995)
3. Coomes, B.A., Koçak, H., Palmer, K.J.: Descrete homoclinic shadowing. Preprint 2001
4. Devaney, R., Nitecki, Z.: Shift automorphismus in the Hénon map. Commun. Math. Phys. **67**, 137–146 (1979)
5. Engel, W.: Ein Satz über ganze Cremona–Transformationen in der Ebene. Math. Ann. **130**, 11–19 (1955)
6. Engel, W.: Ganze Cremona–Transformationen von Primzahlgrad in der Ebene. Math. Ann. **136**, 319–325 (1958)
7. Fontich, E.: Transversal homoclinic points of a class of conservative diffeomorphisms. J. Differ. Equations **87**, 1–27 (1990)
8. Galias, Z., Zgliczynski, P.: Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Hénon map. Nonlinearity **14**, 909–932 (2001)
9. Gelfreich, V., Sauzin, D.: Borel summation and splitting of separatrices for the Hénon map. Ann. Inst. Fourier **51**, 513–567 (2001)
10. Hénon, M.: Numerical study of quadratic area-preserving mappings. Q. Appl. Math. **27**, 291–312 (1969)
11. Hénon, M.: A two–dimensional mapping with a strange attractor. Commun. Math. Phys. **50**, 69–77 (1976)
12. Kirchgraber, U., Manz, U., Stoffer, D.: Rigorous proof of chaotic behaviour in a dumbbell satellite model. J. Math. Anal. Appl. **251**, 897–911 (2000)
13. Kirchgraber, U., Stoffer, D.: Possible chaotic motion of comets in the Sun–Jupiter system – a computer–assisted approach based on shadowing. Nonlinearity **17**, 281–300 (2004)
14. Palmer, K.J.: Exponential dichotomies, the shadowing lemma and transversal homoclinic points. Dynamics Reported 1, pp. 265–306. New York: Wiley Teubner 1988

15. Pilyugin, S.Y.: Shadowing in Dynamical Systems. Lect. Notes Math., vol. 1706. Berlin, Heidelberg, New York: Springer 1999
16. Sterling, D., Dullin, H.R., Meiss, J.D.: Homoclinic bifurcations for the Hénon map. Physica D **134**, 153–184 (1999)
17. Stoffer, D., Palmer, K.: Rigorous verification of chaotic behaviour of maps using validated shadowing. Nonlinearity **12**, 1683–1698 (1999)
18. Stoffer, D., Kirchgraber, U.: Verification of chaotic behaviour in the planar restricted three body problem. Appl. Numer. Math. **39**, 415–433 (2001)
19. Tovbis, A., Tsuchiya, M., Jaffé, C.: Exponential asymptotic expansions and approximations of the unstable and stable manifolds of singularly perturbed systems with the Hénon map as an example. Chaos **8**, 665–681 (1998)